Algorithmic Game Theory, Summer 2025	Lecture 15 (5 pages)

Smooth Mechanisms

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We have spent the past weeks discussing dominant-strategy incentive compatible (truthful) mechanisms. In these mechanisms, for every agent it is always a dominant strategy to report the true value. A classic example is the second-price auction. Today, we will broaden our perspective: What statements can we make if the mechanism is not truthful? For example, if it is a first-price auction?

1 Basic Definitions

Recall our definition of a mechanism-design problem. There is a set N of n players and a set of feasible outcomes X. Every player $i \in N$ has a (private) valuation $v_i \colon X \to \mathbb{R}_{\geq 0}$ from a set of possible valuations V_i. A *mechanism* $\mathcal{M} = (f, p)$ defines a set of bids B_i for each player $i \in N$ and consists of

- an *outcome rule* $f \colon B \to X$, where $B = B_1 \times B_2 \times \cdots \times B_n$, and
- a *payment rule* $p \colon B \to \mathbb{R}^n_{\geq 0}$. So far, we assumed that payments could be arbitrary real numbers. Today, they have to be non-negative.

We say that the mechanism is *direct* if $B_i = V_i$ for all $i \in N$, otherwise we say it is *indirect*. The utility of bidder i on bid profile $b \in B$ is given as $u_i(b, v_i) = v_i(f(b)) - p_i(b)$.

Example 15.1. *We will be considering the following mechanisms.*

- *In a first price auction, we sell one item. Every bidder reports a value. The bidder with the highest reported value wins the item and pays their bid. The other bidders do not pay anything.*
- *An all-pay auction is identical to a first-price auction except for the payments: Every bidder pays their bid, regardless of whether they win or not.*
- *For a more involved example, let us come back to combinatorial auctions. There are m items M, which can each be allocated at most once. Bidders have valuation functions $v_i \colon 2^M \to \mathbb{R}_{\geq 0}$. We can build simple, indirect mechanisms via* item bidding. *Instead of reporting complex functions $2^M \to \mathbb{R}_{\geq 0}$, the bidders now simply report a single bid $b_{i,j}$ for each item j. Each item is sold in a separate first-price or second price-auction. That is, item j is assigned to the bidder i with the highest bid $b_{i,j}$. He has to pay $b_{i,j}$.*

2 Price of Anarchy

For a fixed choice of v, the utilities define a normal-form maximization game. If we assume *complete information*, we study the equilibria of this game. For example, a pure Nash equilibrium is a vector of strategies – in this case bids – such that no player wants to unilaterally deviate.

Definition 15.2 (Pure Nash Equilibrium)**.** *Given a fixed valuation profile $v \in V$, a profile of bids $b = (b_1, \ldots, b_n) \in B$ is a* pure Nash equilibrium *(PNE) if for every player $i \in N$ and every deviation $b'_i \in B_i$,*

$$u_i((b_i, b_{-i}), v_i) \geq u_i((b'_i, b_{-i}), v_i) \ .$$

Also the concepts of mixed Nash and (coarse) correlated equilibria still make sense here.

The goal is to choose an outcome $x \in X$ that maximizes *social welfare* $\sum_{i \in N} v_i(x)$. We use $OPT(v) = \max_{x \in X} \sum_{i \in N} v_i(x)$ to denote the optimal social welfare. For a fixed bid vector b, the mechanism achieves welfare $SW_v(b) = \sum_{i \in N} v_i(f(b)) = \sum_{i \in N} u_i(b, v_i) + \sum_{i \in N} p_i(b)$.

We define the *Price of Anarchy* for any given equilibrium concept as the worst possible ratio between the optimal social welfare and the (expected) social welfare at equilibrium, that is

$$PoA_{\mathsf{Eq}} = \max_{v \in V} \max_{\mathcal{B} \in \mathsf{Eq}(v)} \frac{OPT(v)}{\mathbb{E}_{b \sim \mathcal{B}}[SW_v(b)]},$$

where $\mathsf{Eq}(v)$ denotes the set of equilibria for the game induced by valuations v.

This ratio is always at least 1 because the optimal social welfare can never be smaller than the social welfare in equilibrium. Ratios closer to 1 are better. Furthermore, we have again

$$1 \leq PoA_{\mathsf{PNE}} \leq PoA_{\mathsf{MNE}} \leq PoA_{\mathsf{CCE}} \ .$$

3 Price of Anarchy of First-Price Auction

Let's first consider a first-price auction and bound the price of anarchy.

Theorem 15.3. *In a first-price auction, $PoA_{\mathsf{PNE}} \leq 2$.*

Proof. Observe, that we have for all $i \in N$

$$u_i\left(\left(\frac{v_i}{2}, b_{-i}\right), v_i\right) \geq \frac{v_i}{2} - \max_{i'} b_{i'} \quad \text{and} \quad u_i\left(\left(\frac{v_i}{2}, b_{-i}\right), v_i\right) \geq 0 \ .$$

The first inequality follows by a simple case distinction: Either i wins the item with bid $\frac{v_i}{2}$, then the utility is $v_i - \frac{v_i}{2} = \frac{v_i}{2}$. Or i loses, so then $\max_{i'} b_{i'} \geq \frac{v_i}{2}$. The second inequality follows because in any case the utility is non-negative. In combination, this gives us for any v and any b

$$\sum_{i \in N} u_i\left(\left(\frac{v_i}{2}, b_{-i}\right), v_i\right) \geq \max_{i \in N} u_i\left(\left(\frac{v_i}{2}, b_{-i}\right), v_i\right) \geq \max_{i \in N} \frac{v_i}{2} - \max_{i \in N} b_i = \frac{1}{2} OPT(v) - \sum_{i \in N} p_i(b) \ . \tag{1}$$

To bound the social welfare $\sum_{i \in N} v_i(f(b))$, it is is important to observe that we can also write it as the sum of utilities and payments: $\sum_{i \in N} v_i(f(b)) = \sum_{i \in N} u_i(b, v_i) + \sum_{i \in N} p_i(b)$.

By using Equation (1), we get that if b is a pure Nash equilibrium $SW_v(b) = \sum_{i \in N} u_i(b, v_i) + \sum_{i \in N} p_i(b) \geq \sum_{i \in N} u_i((\frac{v_i}{2}, b_{-i}), v_i) + \sum_{i \in N} p_i(b) \geq \frac{1}{2} OPT(v)$. □

4 The Smoothness Framework

We define smooth mechanisms and show how smoothness implies that all equilibria of a mechanism are close to optimal.

Definition 15.4 (Smooth Mechanism, simplified version)**.** *Let $\lambda, \mu \geq 0$. A mechanism $\mathcal{M}$ is (λ, μ)-smooth if for any valuation profile $v \in V$ for each player $i \in N$ there exists a bid b_i^* such that for any profile of bids $b \in B$ we have*

$$\sum_{i \in N} u_i((b_i^*, b_{-i}), v_i) \geq \lambda \cdot OPT(v) - \mu \sum_{i \in N} p_i(b) \ .$$

Note that, by the order of the quantifiers, b_i^* may depend on the profile of valuations but not on the bids. Equation (1) already shows that the single-item first-price auction is $(1/2, 1)$ smooth by setting $b_i^* = \frac{v_i}{2}$.

Observation 15.5. *A single-item first-price auction is* $(1/2, 1)$*-smooth.*

As another example, let us consider the single-item all-pay auction.

Theorem 15.6. *A single-item all-pay auction is* $(\frac{1}{2}, 2)$*-smooth.*

Proof. Let j be a bidder with highest value v_j. Set $b_j^* = v_j/2$ and $b_i^* = 0$ for $i \neq j$. Consider an arbitrary bid profile $b \in B$.

We show that always $u_j((b_j^*, b_{-j}), v_j) \geq \frac{1}{2}v_j - 2\max_{i \neq j} b_i$. We distinguish two cases. If bidder j wins the item in (b_j^*, b_{-j}), then his utility is $v_j - b_j^* = \frac{1}{2}v_j$. So, the bound is fulfilled because bids are non-negative. If he does not win the item, then his utility is $-\frac{1}{2}v_j$. As he loses, somebody must outbid him, meaning that $\max_{i \neq j} b_i \geq \frac{1}{2}v_j$. So, the bound holds as well.

Furthermore, by non-negativity of bids, $\sum_i p_i(b) = \sum_i b_i \geq \max_{i \neq j} b_i$.

Finally, for all $i \neq j$, we have $u_i((b_i^*, b_{-i}), v_i) \geq 0$ because $b_i^* = 0$ and therefore regardless of b_{-i} the bidder does not have to pay anything.

In combination, this gives us

$$\sum_i u_i((b_i^*, b_{-i}), v_i) \geq u_j((b_j^*, b_{-j}), v_j) \geq \frac{1}{2}v_j - 2\max_i b_i \geq \frac{1}{2}v_j - 2\sum_i p_i(b) = \frac{1}{2}OPT(v) - 2\sum_i p_i(b) \ .$$

Therefore, this auction is $(\frac{1}{2}, 2)$-smooth. □

5 Price-of-Anarchy Bound

Smoothness implies a bound on the price of anarchy. The proof works just like for smooth games. To keep things simple, we consider pure Nash equilibria only.

Theorem 15.7 (Syrgkanis and Tardos, 2013)**.** *If a mechanism* $\mathcal{M}$ *is* (λ, μ)*-smooth and players have the possibility to withdraw from the mechanism then*

$$PoA_{\mathsf{PNE}} \leq \frac{\max\{\mu, 1\}}{\lambda} \ .$$

Proof. Suppose bid profile b is a pure Nash equilibrium. This means that no player wants to unilaterally deviate from the equilibrium bid to some other bid. That is,

$$u_i((b_i, b_{-i}), v_i) \geq u_i((b_i', b_{-i}), v_i) \ ,$$

for all players $i \in N$ and bids $b_i' \in B_i$.

Now in particular players do not want to deviate to the bid b_i^* whose existence is guaranteed by smoothness. Considering, for each player $i \in N$ the deviation to b_i^* and summing over all players,

$$\sum_{i \in N} u_i((b_i, b_{-i}), v_i) \geq \sum_{i \in N} u_i((b_i^*, b_{-i}), v_i) \geq \lambda \cdot OPT(v) - \mu \cdot \sum_{i \in N} p_i(b) \ .$$

Since players have quasi-linear utilities $u_i(b, v_i) = v_i(f(b)) - p_i(b)$ or $v_i(f(b)) = u_i(b, v_i) + p_i(b)$. Using this we obtain

$$\sum_{i \in N} v_i(f(b)) \geq \lambda \cdot OPT(v) + (1 - \mu) \cdot \sum_{i \in N} p_i(b) \ .$$

Notice that the left-hand side is precisely the social welfare at equilibrium. So if $\mu \leq 1$ we can bound $(1 - \mu) \cdot \sum_{i \in N} p_i(b) \geq 0$ and obtain

$$\sum_{i \in N} v_i(f(b)) \geq \lambda \cdot OPT(v) \ ,$$

which shows a Price of Anarchy of $1/\lambda = \max\{1, \mu\}/\lambda$.

On the other hand, if $\mu > 1$, we can use that players have the right to withdraw from the mechanism and obtain a utility of zero to argue that $u_i(b, v_i) = v_i(f(b)) - p_i(b) \geq 0$ and so $p_i(b) \leq v_i(f(b))$. Since $(1 - \mu) < 0$ we obtain

$$\sum_{i \in N} v_i(f(b)) \geq \lambda \cdot OPT(v) + (1 - \mu) \cdot \sum_{i \in N} v_i(f(b)) \enspace .$$

Subtracting $(1 - \mu) \cdot \sum_{i \in N} v_i(f(b))$ and dividing by $\mu > 1$ we obtain

$$\sum_{i \in N} v_i(f(b)) \geq \lambda/\mu \cdot OPT(v) \enspace ,$$

which again shows a Price of Anarchy bound of $\mu/\lambda = \max\{1, \mu\}/\lambda$. □

This argument extends to more general equilibrium concepts such as coarse correlated equilibria. The only point where we used the equilibrium condition is when we argued that players do not want to deviate from the equilibrium bid b_i to some other bid b'_i. In fact, the specific deviations that we considered only depended on the valuation profile v and did not depend on the bids b. Hence the exact same argument applies to coarse correlated equilibria and shows an upper bound on the Price of Anarchy of $\max\{1, \mu\}/\lambda$.

6 Item Bidding

Finally, we come to combinatorial auctions with item bidding. Recall *unit-demand valuations*: These are functions v_i such that there are $v_{i,j} \in \mathbb{R}_{\geq 0}$ such that $v_i(S) = \max_{j \in S} v_{i,j}$. So, a bidder does not get any further value from more than one item. However, in item bidding, a bidder can potentially win multiple items, even if they only want one. More concretely, if, for example, $v_{i,1} = \ldots = v_{i,m} = 1$, then bidder i has a value of 1 as long as he receives an item, no matter which. There is no way to express this in a bid. Therefore, this is not a direct mechanism and it cannot be truthful. However, its Price of Anarchy is bounded by 2.

Theorem 15.8. *For unit-demand valuations, item bidding with first-price payments is* $(\frac{1}{2}, 1)$*-smooth.*

Proof. We have to devise the deviation bids b_i^* for all bidders. These bids may depend on the valuations v but not on the bids. Consider the welfare-maximizing allocation on v. Let j_i be the item that is assigned to bidder i in this allocation. If i does not get any item, set j_i to $\perp$.

We now set $b_{i,j}^* = \frac{v_{i,j}}{2}$ if $j = j_i$ and 0 otherwise. That is, in the deviation bid, each bidder bids half his value on the item that he is supposed to get.

Given any bid profile b, bidder i's utility after deviating is $\frac{v_{i,j_i}}{2}$ unless another bidder bids at least $\frac{v_{i,j_i}}{2}$ for item j_i in b. Therefore

$$u_i((b_i^*, b_{-i}), v_i) \geq \frac{v_{i,j_i}}{2} - \max_{i' \neq i} b_{i',j_i} \geq \frac{v_{i,j_i}}{2} - \max_{i'} b_{i',j_i} \enspace .$$

If we take the sum over all bidders i, then

$$\sum_{i \in N} u_i((b_i^*, b_{-i}), v_i) \geq \sum_{i \in N} \frac{v_{i,j_i}}{2} - \sum_{i \in N} \max_{i'} b_{i',j_i} \enspace .$$

Observe that $\sum_{i \in N} v_{i,j_i} = OPT(v)$ because of the way we defined j_i. Furthermore, we have $\sum_{i \in N} \max_{i'} b_{i',j_i} \leq \sum_{j \in M} \max_{i'} b_{i',j} = \sum_{i \in N} p_i(b)$ because every item is counted at most once: For each item j there is at most one i such that $j = j_i$. That is,

$$\sum_{i \in N} u_i((b_i^*, b_{-i}), v_i) \geq \frac{1}{2} OPT(v) - \sum_{i \in N} p_i(b) \enspace ,$$

which is exactly $(\frac{1}{2}, 1)$-smoothness. □

So, immediately we get that the Price of Anarchy for pure Nash equilibria is at most 2.

7 Second-Price Auctions

Our results today were for the first-price auction, the all-pay auction, and generalizations thereof. Maybe it would be more natural to generalize the second-price auction. In the case of item bidding this would mean that each item is sold in a separate second-item auction. However, there are some issues as we see in this example.

Example 15.9. *Consider a single-item second-price auction with two bidders. Let $\epsilon > 0$ be small. Then for $v_1 = 1$, $v_2 = \epsilon$, it is a pure Nash equilibrium $b_1 = 0$, $b_2 = 1$. Here the second bidder pays nothing and wins the item. The first bidder does not want to bid more because they would have to pay at least* 1 *to win the item. So, the Price of Anarchy is unbounded.*

One can indeed get bounds on the Price of Anarchy when assuming that bidders do not overbid. See the referenced papers for more details.

References and Further Reading

- Vasilis Syrgkanis and Éva Tardos. Composable and Efficient Mechanisms. STOC'13. (Smoothness for mechanisms)
- Paul Dütting and Thomas Kesselheim. Algorithms against Anarchy: Understanding Non-Truthful Mechanisms. EC'15. (Characterization of algorithms with small PoA)
- George Christodoulou, Annamária Kovács, Michael Schapira: Bayesian Combinatorial Auctions. J. ACM 63(2): 11:1-11:19 (2016) (The first paper on item bidding.)